Orbiter Technical Notes: Nonspherical gravitational field perturbations

Martin Schweiger

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1 Introduction

Orbiter uses a zonal representation of the gravitational potential generated by a celestial body, using a Legendre polynomial series expansion in the latitude θ. The perturbations in longitude (ϕ) are assumed to be negligible. The potential is expressed as

$$U_G(r,\phi,\theta) = -\frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\theta)\right] \tag{1}$$

where G is the gravitational constant, M and R are the mass and mean radius of the central body, respectively, r is the length of the radius vector, J_n are the coefficients of the series expansion, and P_n are the Legendre polynomials of order n. The first Legendre polynomials are defined as

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned} \tag{2}$$

The acceleration due to the gravitational field of a test mass at point $\vec{r} = (r,\phi,\theta)$ is then given by the gradient of the potential:

$$\vec{a}_G(r,\phi,\theta) = -\vec{\nabla} U_G(r,\phi,\theta) \tag{3}$$

In spherical polar coordinates, the gradient operator is expressed as

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\cos\theta}\hat{\phi}\frac{\partial}{\partial\phi} \tag{4}$$

Substituting equations 1 and 4 into 3 yields

$$\vec{a}_G(r,\phi,\theta) = \hat{r}a_0^{(r)}(r) - \sum_{n=2}^{\infty}\left[\hat{r}a_n^{(r)}(r,\theta) + \hat{\theta}a_n^{(\theta)}(r,\theta)\right] \tag{5}$$

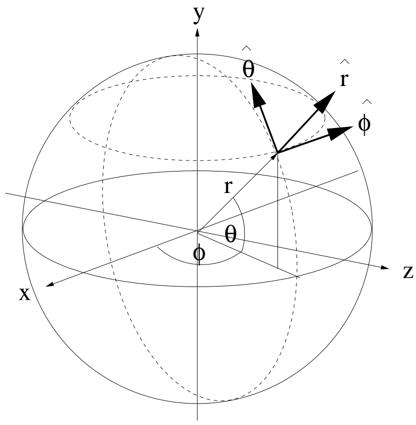

Figure 1: Planet-relative coordinates and polar unit vectors at a point (r, ϕ, θ).

	J_2	J_3	J_4	J_5
Mercury	60	-	-	-
Venus	27	-	-	-
Earth	1082.6269	-2.51	-1.60	-0.15
Mars	1964	-	-	-
Jupiter	14750	-	-	-
Saturn	16450	-	-	-
Uranus	12000	-	-	-
Neptune	4000	-	-	-

Table 1: Coefficients ($\times 10^6$) for zonal expansion of planetary gravitational potentials.

with the first terms given by

$$\begin{aligned}
a_0^{(r)}(r) &= -\frac{GM}{r^2} \\
a_2^{(r)}(r,\theta) &= -\frac{3}{2}\frac{GMR^2J_2}{r^4}(3\sin^2\theta - 1) \\
a_2^{(\theta)}(r,\theta) &= 3\frac{GMR^2J_2}{r^4}\sin\theta\cos\theta \\
a_3^{(r)}(r,\theta) &= -2\frac{GMR^3J_3}{r^5}(5\sin^3\theta - 3\sin\theta) \\
a_3^{(\theta)}(r,\theta) &= \frac{3}{2}\frac{GMR^3J_3}{r^5}(5\sin^2\theta\cos\theta - \cos\theta) \\
a_4^{(r)}(r,\theta) &= -\frac{5}{8}\frac{GMR^4J_4}{r^6}(35\sin^4\theta - 30\sin^2\theta + 3) \\
a_4^{(\theta)}(r,\theta) &= \frac{5}{2}\frac{GMR^4J_4}{r^6}(7\sin^3\theta\cos\theta - 3\sin\theta\cos\theta)
\end{aligned} \tag{6}$$

The coefficients J_n used by Orbiter are listed in Table 1.

The field perturbations can lead to a rotation of the orbit trajectory of a satellite. This rotation can be expressed in terms of the movement of the longitude of the ascending node (Ω) and the movement of the argument of periapsis (ω). If only terms up to J_2 are included, approximate values of the movements $\partial\Omega/\partial t$ and $\partial\omega/\partial t$ are given by

$$\frac{\partial\Omega}{\partial t} = -\frac{3n}{2}\left(\frac{R}{a}\right)^2 \frac{\cos i}{(1-e^2)^2} J_2 \tag{7}$$

$$\frac{\partial\omega}{\partial t} = \frac{3n}{4}\left(\frac{R}{a}\right)^2 \frac{5\cos^2 i - 1}{(1-e^2)^2} J_2 \tag{8}$$

where $n = 2\pi/P$ is the mean motion (with orbit period P), a is the mean distance, e is the eccentricity, and i is the inclination.

Example: calculate the inclination for a sun-synchronous polar orbit

A sun-synchronous orbits exploits the propagation of the line of nodes to keep the orbital plane synchronised with the relative position of the sun. A satellite can for example be placed in a sun-synchronous orbit so that it continuously flys over the planet's terminator line. From 7 we have

$$\cos i = -\frac{2}{3n}\left(\frac{a}{R}\right)^2 \frac{(1-e^2)^2}{J_2}\frac{\partial\Omega}{\partial t} \tag{9}$$

A sun-synchronous orbit requires the line of nodes to move at a rate of 2π per year. For Earth, this is equivalent to $\partial\Omega/\partial t = 1.99 \cdot 10^{-7}$ rad/s (about 0.99 deg. per day). Assume a circular orbit ($e = 0$) at an altitude of 300 km ($a = 6671010$ m, with $R_E = 6371010$ m). With $P = 2\pi\sqrt{a^3/\mu_E}$ we get $n = \sqrt{\mu_E/a^3} = 0.0012$ rad/s. This leads to

$$\cos i_{\text{sync}} = -\frac{2}{0.0035}\left(\frac{6678137}{6378137}\right)^2 \frac{1.99 \cdot 10^{-7}}{0.001082630} = -0.116, \tag{10}$$

or $i_{\text{sync}} = 96.7$ deg.